Theorem 1:

Any Clifford operations, applied to the input state $|0\rangle^{\otimes n}$ followed by the Z measurements, can be simulated efficiently in the strong sense.

Proof:

The stabilizer group of the input state is $\langle \{Z_i\} \rangle$ $(i = 0, 1, \dots, n-1)$

$$\xrightarrow{\text{Clifford operations}} \langle \{S_i\} \rangle$$

suppose measurement output is given by $\{m_i = 0, 1\}$ $\longrightarrow$ probability can be calculated as follows:

(i) set stabilizer generators $\mathcal{S}^{(0)} = \langle \{S_i\} \rangle$ and initial probability $p^{(0)} = 1$

(ii) For $k = 0, 1, \dots, n-1$, repeat the following

1) If $(-1)^{m_k} Z_k \in \mathcal{S}^{(k)}$, update the probability $p^{(k+1)} = p^{(k)}$, because the measurement outcome m_k is obtained

with probability 1.

update stabilizer group $\mathcal{S}^{(k+1)} = \mathcal{S}^{(k)}$

2) Else, if $(-1)^{m_k \oplus 1} Z_k \in \mathcal{S}^{(k)}$

$\longrightarrow$ update $p^{(k+1)} = 0$

3) Else, $\mathcal{S}^{(k)}$ is updated into $\mathcal{S}^{(k+1)}$ by removing anticommuting generator and adding $(-1)^{m_k} Z_k$ as new generator

update $p^{(k+1)} = p^{(k)}/2$

iii) Return $p^{(n)}$ as the probability of obtaining measurement outcome $\{m_i\}$.

<u>Note</u>: Can efficiently decide about the occurence of the 3 cases in ii) by checking commutability of Z_k with stabilizer generators of $\mathcal{S}^{(k)}$.

$\square$

Theorem 2.2:

Any Clifford operations, applied to any product states of convex mixtures of the Pauli basis states, followed by Z measurements can be efficiently simulated in the weak sense.

"weak sense": classical simulation of a quantum circuit which measures the output x according to prob. distr. $P_c(x)$ (without explicit computation of $P_c(x)$)

Proof:

Suppose the ith input qubit is given by

$$\rho_i = p_{x,+}^{(i)} |+\rangle\langle+| + p_{x,-}^{(i)} |-\rangle\langle-| + p_{y,+}^{(i)} |+i\rangle\langle+i| + p_{y,-}^{(i)} |-i\rangle\langle-i| + p_{z,+}^{(i)} |0\rangle\langle 0| + p_{z,-}^{(i)} |1\rangle\langle 1|$$

where $\sum_{\alpha=x,y,z} \sum_{\nu=\pm} p_{\alpha,\nu}^{(i)} = 1$

$\longrightarrow$ input state of each qubit randomly sampled $\{p_{\alpha,\nu}^{(i)}\}$

Theorem 2.1 $\rightarrow$ for each product of Pauli basis states, output prob. distr. can be calculated

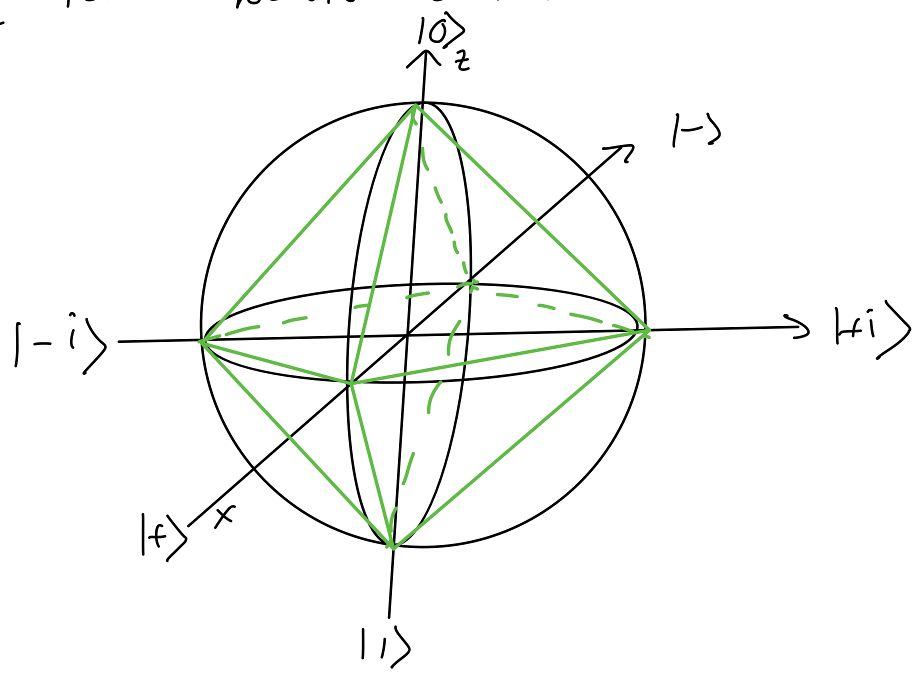
$\rightarrow$ combine with random sampling $\{p^{(i)}_{\alpha,\nu}\}$

$\square$

<u>Note</u>:

1) The $e^{-i(\pi/8)Z}$-gate is a non-Clifford gate $\rightarrow$ no universal quantum comp.

2) $e^{-i(\pi/8)Z}|+\rangle$ is a non-stabilizer state, and lies outside the convex mixture of Pauli basis states



§ 2.5 Graph States

- defined by a graph $G=(V,E)$

  vertices   edges

- On each vertex there is a qubit
- stabilizer generator of graph state $|G\rangle$ is defined as

$$K_i = X_i \prod_{j\in V_i} Z_j \qquad \forall\ i\in V$$

where $V_i := \{ j \mid (i,j)\in E \}$

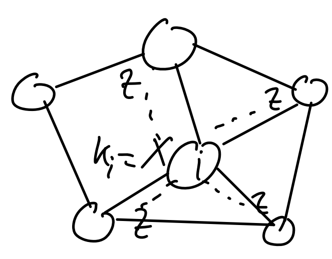
- $|G\rangle$ is defined by

$$|G\rangle = \prod_{(i,j)\in E} \Lambda(Z)_{i,j} \, |+\rangle^{\otimes |V|}$$

CZ-gate

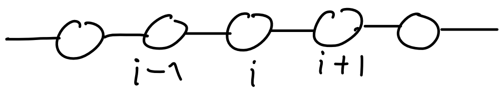
(X_i is transformed to K_i by $\prod \Lambda(Z)_{i,j}$)

regular lattices → cluster states



Pauli basis measurements:

consider 1D graph state



$\longrightarrow$ stabilizer generator: $K_i = Z_{i-1} X_i Z_{i+1}$

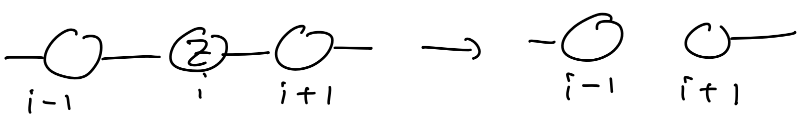
1) measurement of qubit i along Z-basis leads to:

$$\langle \ldots, K_{i-1}, K_i, K_{i+1}, \ldots \rangle$$
$$\downarrow$$
$$\langle \ldots, K_{i-1}, Z_i, K_{i+1}, \ldots \rangle$$

ith qubit is $|0\rangle$

$\rightarrow$ 3 decoupled stabilizer groups

$$\langle \ldots, Z_{i-2} X_{i-1} \rangle, \langle Z_i \rangle, \langle X_{i+1} Z_{i+2}, \ldots \rangle$$



$\rightarrow$ this splitting holds for any graph

2) X-basis measurement:

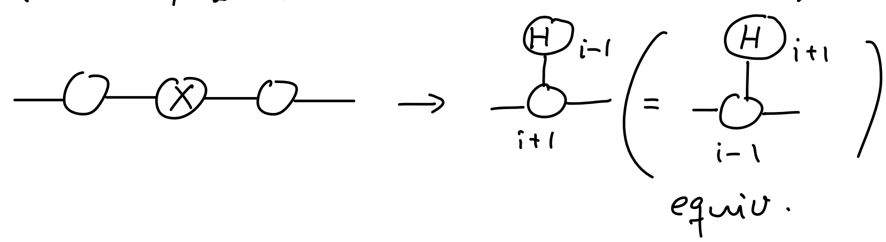
$[X_i, K_{i-1} K_{i+1}] = 0$, $K_{i-1} K_{i+1} = Z_{i-2} X_{i-1} X_{i+1} Z_{i+2}$

$\longrightarrow$ post-measurement stabilizer group:

$$\langle \ldots, Z_{i-2} X_{i-1} X_{i+1} Z_{i+2}, Z_{i-1} Z_{i+1}, \ldots \rangle, \langle X_i \rangle$$

Hadamard operation on $(i-1)$th qubit

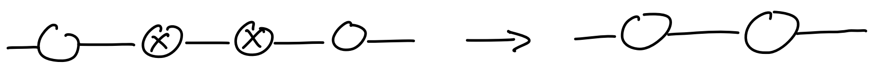
$\rightarrow \langle \dots, Z_{i-2} Z_{i-1} X_{i+1} Z_{i+2}, X_{i-1} Z_{i+1}, \dots \rangle, \langle X_i \rangle$



(*)

measurement of ith and $(i+1)$th qubit in X-basis $\rightarrow$ equiv to measuring $(i+1)$th qubit in (*)

1) $\rightarrow$ remove $(i+1)$th node from graph



3) Consider Y-basis measurement

Y_i commutes with $K_{i-1} K_i = Z_{i-2} Y_{i-1} Y_i Z_{i+1}$

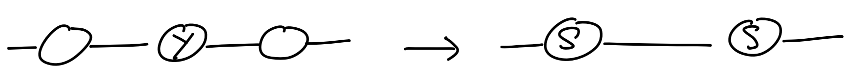
and $K_i K_{i+1} = Z_{i-1} Y_i Y_{i+1} Z_{i+2}$

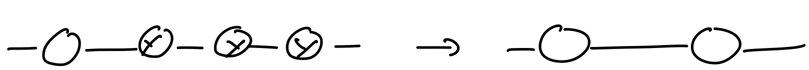
$\rightarrow$ post-measurement stabilizer group

$\langle \dots, Z_{i-2} Y_{i-1} Z_{i+1}, Z_{i-1} Y_{i+1} Z_{i+2}, \dots \rangle, \langle Y_i \rangle$

phase S on $(i-1)$th and $(i+1)$th qubit:

$\langle \dots, Z_{i-2} X_{i-1} Z_{i+1}, Z_{i-1} X_{i+1} Z_{i+2}, \dots \rangle, \langle Y_i \rangle$



also: 

§ 2.6 Measurement-Based Quantum Comp.

1) Quantum teleportation:

Suppose Alice and Bob share a maximally entangled Bell-state:

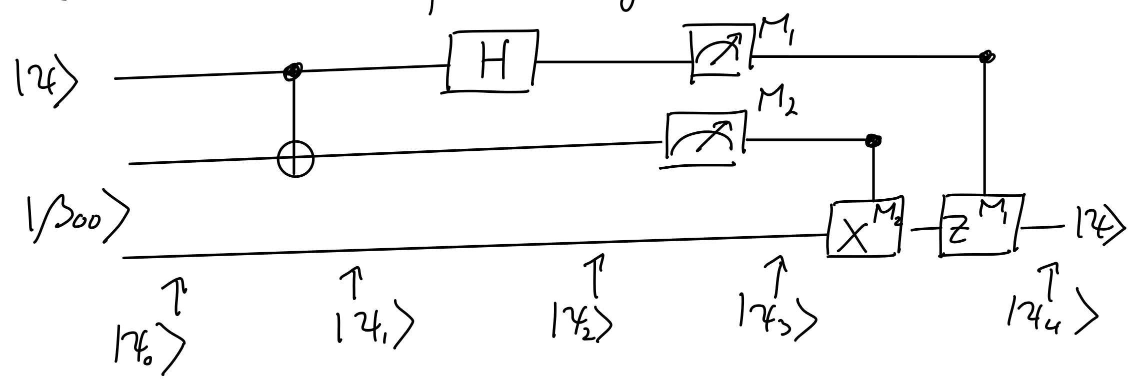
$$|\beta_{00}\rangle = \frac{|0\rangle_a|0\rangle_b + |1\rangle_a|1\rangle_b}{\sqrt{2}}$$

Now consider an arbitrary qubit

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

with α and β unknown to Alice.

Consider the following circuit:



We start with

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)\right]$$

as input state

CNOT $\longrightarrow$ $|\psi_1\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|10\rangle + |01\rangle)\right]$

Hadamard $\rightarrow$ $|\psi_2\rangle = \frac{1}{2}\Big[\alpha(|0\rangle + |1\rangle)(|00\rangle + |11\rangle)$

$\qquad + \beta(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)\Big]$

Regrouping gives

$$|\psi_2\rangle = \frac{1}{2}\Big[|00\rangle(\alpha|0\rangle+\beta|1\rangle) + |01\rangle(\alpha|1\rangle+\beta|0\rangle) + |10\rangle(\alpha|0\rangle-\beta|1\rangle) + |11\rangle(\alpha|1\rangle-\beta|0\rangle)\Big]$$

Bob's post-measurement state:

$$00 \longmapsto |\psi_3(00)\rangle \equiv \left[\alpha|0\rangle+\beta|1\rangle\right]$$
$$01 \longmapsto |\psi_3(01)\rangle \equiv \left[\alpha|1\rangle+\beta|0\rangle\right]$$
$$10 \longmapsto |\psi_3(10)\rangle \equiv \left[\alpha|0\rangle-\beta|1\rangle\right]$$
$$11 \longmapsto |\psi_3(11)\rangle \equiv \left[\alpha|1\rangle-\beta|0\rangle\right]$$

Now Bob just has to apply $X^{M_2}Z^{M_1}$ for $M_i = (0,1)$ to his qubit to recover $|\psi\rangle$! Voila!